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LETTER TO THE EDITOR

Possibly exact fractal dimensions from conformal invariance

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Abstract. Based on an equivalence between the q-state Potts model and conformally invariant field theories, we speculate that all fractal dimensionalities of two-dimensional percolation are given by the formula $D=(100-x^2)/48$, where x is an integer. The same formula should also apply to self-avoiding walks and lattice animals. This approach is extended to cluster-weighted percolation. Our conjecture seems to agree with previous exact and numerical results except for the backbone exponent, and possibly for lattice animals.

It has long been known that percolation corresponds to the $q=1$ limit of the q-state Potts model (Kasteleyn and Fortuin 1969). In particular, the fractal dimensionalities of the incipient infinite cluster equals the magnetic scaling power, $D_f=y_h=\frac{91}{48}$ in two dimensions (Stanley 1977), while that of the singly connected ('red') bonds is given by the thermal exponent, $D_{\text{red}}=y_t=\frac{3}{4}$ (Coniglio 1981, 1982). The dimensionalities of several other subsets of the percolation cluster have been measured, such as that of the backbone (D_{BB}), the minimal path or chemical distance ($D_{\min}$), the hull (D_{H}), the unscreened (external) perimeter (D_{u}), the elastic backbone or collection of minimal paths (D_{EBB}) and the internal dangling bond (D_{I}). For the definitions of these and other subsets, see e.g. Stanley (1984) and Grossman and Aharony (1986).

Recently, a new approach to 2D spin models has been initiated, based on conformal invariance (Belavin *et al* 1984a, b, Dotsenko and Fateev 1984, Friedan *et al* 1984). The key observation is that a spin model at the critical temperature is not only invariant under global scale transformations, but also under local, conformal transformations, which change length scales locally but preserve angles (Polyakov 1970). In two dimensions, conformal invariance is sufficient to uniquely determine all critical exponents, as soon as one has identified the universality class of the model. The universality classes can be labelled by one real number, the central charge c. For so-called minimal theories (including a countably infinite set of Potts models), the conformal dimensions are given by the Kac formula (Kac 1979, Dotsenko and Fateev 1984)

$$\Delta_{nm}=\frac{(pn-km)^2-(k-p)^2}{4pk} \tag{1}$$

where

$$k=2N \qquad p=2N-1$$

and (Temperley and Lieb 1971, Baxter *et al* 1976, Dotsenko and Fateev 1984)

$$q^{1/2}=2\cos(\pi/2N) \qquad 1\leqslant N<\infty.$$

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As usual, q is the number of Potts states. The corresponding value of the central charge is

$$c = 1 - 6/2N(2N-1).$$

In particular, percolation corresponds to $q = 1$, $N = \frac{3}{2}$ and $c = 0$. Since percolation in this way is closely related to a conformal field theory, we conjecture that the fractal dimensionalities of all subsets of the percolation cluster follow from (1), with $N = \frac{3}{2}$. In two dimensions, a fractal dimension D is related to a conformal dimension Δ by

$$D = 2 - 2\Delta$$

because all correlation functions at the critical point decay as

$$G(r) \sim r^{-2(2-D)} \sim r^{-4\Delta}.$$

For minimal theories, which have N integer, the parameters n and m in the Kac formula (1) are integers. This gives only a few possibilities for the conformal dimensions. For non-minimal theories, such as $N = \frac{3}{2}$, no such statement can be made. However, we have recently investigated an interacting percolation problem, cluster-weighted percolation (CWP) (Larsson 1986, Hu 1984), which is a percolation formulation of the Potts model for arbitrary q. We can define the same subsets of the cluster-weighted percolation cluster as in ordinary percolation. If we assume that the dimensionalities for these fractals depend smoothly and simply on N, n and m must be functions of N which reduce to integers for N integer. Specifically, we will consider

$$\begin{aligned} &n, m = 1, 2, 3 \\ &n, m = 2N, 2N-1, 2N-2, 2N-3 \\ &n, m = N, N \pm 1, N \pm 2, N \pm 3. \end{aligned} \tag{2}$$

What are the consequences of this conjecture? Setting $k = 3$, $p = 2$, $n = i/2$ and $m = j/2$ in (1), we obtain

$$\Delta = (x^2 - 4)/96$$

or

$$D = 2 - 2\Delta = (100 - x^2)/48 \tag{3}$$

where $x = 2i - 3j$ is an integer. The possible values for D (with $0 \leqslant D \leqslant 2$) are given in table 1. In particular, for $x = 3$ we get $D = \frac{91}{48} = D_f$, and for $x = 8$, $D = \frac{3}{4} = D_{red}$.

In fact, there are other well known models which correspond to $N = \frac{3}{2}$. The O(n) model (at least on a honeycomb lattice) can be parametrised by (Nienhuis 1982, Dotsenko and Fateev 1984)

$$n = -2\cos\frac{2\pi}{2 - 1/N} = 2\cos\frac{\pi}{2N-1} \qquad 1 \leqslant N < \infty.$$

Table 1. Possible fractal dimensionalities for percolation.

x	2	3	4	5	6	7	8	9	10
D	2	$\frac{91}{48}$	$\frac{7}{4}$	$\frac{25}{16}$	$\frac{4}{3}$	$\frac{51}{48}$	$\frac{3}{4}$	$\frac{19}{48}$	0
D_{approx}	2	1.896	1.75	1.5625	1.333	1.0625	0.75	0.396	0

Thus $N=\frac{3}{2}$ corresponds to $n=0$, i.e. self-avoiding walks (SAW) or linear polymers (de Gennes 1972). The dimension $D_{\mathrm{SAW}}=\frac{4}{3}\approx 1.33$ follows from (3) if $x=6$. Since both percolation and SAW correspond to the same conformal field theory, we believe that this is the case also for the third purely geometrical model we are aware of, namely branched polymers or lattice animals (LA). This conjecture is corroborated by the fact that a model can be defined which yields both percolation and LA in different limits (Harris and Lubensky 1981). The best estimate of the LA dimension, $D_{\mathrm{LA}}^{-1}\approx 0.6406\pm 0.0001$ (Kertész 1986) is very close to our prediction for $x=5$, $D=\frac{25}{16}=1.5625$ or $D^{-1}=0.6400$. Unfortunately, the discrepancy seems to be somewhat too large to be attributable to statistical errors.

In table 2 we list estimates for the fractal dimensions of the different subsets, together with the value from (3) which gives the best correspondence. There are three discrepancies.

(i) Most workers obtain a $D_{\min}$ in the range 1.1–1.2. This seems to be in contradiction with the only possibility from (3), $D_{\min}=\frac{17}{16}=1.0625$. However, Edwards and

Table 2. Comparison of our conjecture (3) with previously obtained numerical estimates (floating-point numbers) and exact and conjectured values (rational numbers).

Subset	D measured	D conjectured
Percolation cluster		$\frac{91}{48}\approx 1.895$
Stanley (1977)	$\frac{91}{48}$	
Red bonds		$\frac{3}{4}=0.75$
Coniglio (1981, 1982)	$\frac{3}{4}$	
Hull		$\frac{7}{4}=1.75$
Voss (1984)	1.74 ± 0.02	
Sapoval *et al* (1985)	$\frac{7}{4}$ (conjectured)	
Grassberger (1986)	1.750 ± 0.002	
Ziff (1986)	1.751 ± 0.002	
Unscreened perimeter		$\frac{4}{3}\approx 1.333$
Grossman and Aharony (1986)	1.37 ± 0.03	
Internal dangling bonds		$\frac{7}{4}=1.75$
Grossman and Aharony (1986)	1.77 ± 0.04	
Minimal path		1?
Alexandrowicz (1980)	1.18 ± 0.08	$\frac{17}{16}=1.0625$?
Pike and Stanley (1981)	1.12 ± 0.02	
Grassberger (1983)	1.12 ± 0.02	
Grassberger (1985)	1.13 ± 0.01	
Hong and Stanley (1983)	1.04 ± 0.1	
Havlin and Nossal (1984)	1.16 ± 0.04	
Herrmann *et al* (1984)	1.10 ± 0.06	
Edwards and Kerstein (1985)	1.021 ± 0.005	
Backbone		$\frac{25}{16}=1.5625$?
Herrmann *et al* (1984)	1.60 ± 0.05	$\frac{7}{4}=1.75$?
Herrmann and Stanley (1984)	1.62 ± 0.02	
Puech and Rammal (1983)	1.68 ± 0.02	
Elastic backbone		$D_{\min}$
Herrmann *et al* (1984)	1.10 ± 0.05	
SAW		$\frac{4}{3}\approx 1.333$
Nienhuis (1982)	$\frac{4}{3}$	
Lattice animals		$\frac{25}{16}=1.5625$
Derrida and de Sèze (1982)	1.561 ± 0.001	
Kertész (1986)	1.5610 ± 0.0002	

Kerstein (1985) have recently obtained a considerably lower value, using a different approach to overcome finite-size effects. Even though their $D_{\min} \approx 1.02$ is a bit too small, it shows that $D_{\min} = \frac{17}{16}$ cannot be ruled out. Based on arguments given below, however, we suspect that the minimal path is not a fractal at all, but simply a curve of finite length, and thus $D_{\min} = 1$.

(ii) The estimates for $D_{\rm BB}$ lie in the range 1.6-1.7, which is inconsistent with (3), which only yields the possibilities $D = \frac{7}{4} = 1.75$ and $D = \frac{25}{16} = 1.5625$. This is a considerable disappointment, because the backbone is such a conceptually simple subset which should correspond to a conformal field. However, it is notoriously difficult to measure (Stanley *et al* 1986). Perhaps some finite-size corrections along the lines of Edwards and Kerstein could bring down the value to $D_{\rm BB} = \frac{25}{16} = 1.5625$, but we find this most unlikely.

(iii) The dimensionality of doubly connected bonds (such that breaking both bonds in a pair cuts the fractal in two) is $D_{\rm double} = \frac{3}{2} = 1.5$ (Aharony *et al* 1986). This exact result does not agree with (3), which shows that care must be taken. The only explanation is that the doubly connected bonds do not correspond to a conformal field. Perhaps this fractal can be viewed as some sort of square of the red bonds. This would also explain why Coniglio's (1981, 1982) proof could be extended to this fractal.

As mentioned above, the q-state Potts model for arbitrary q can be viewed as an interacting percolation problem (CWP). The derivation of this equivalence by Larsson (1986) is incorrect, but the conclusion is still valid, which was previously shown by Hu (1984). There is no conceptual problem in defining the same subsets of a CWP cluster as in ordinary percolation. Also the $O(n)$ model permits a geometrical interpretation for arbitrary n, the loop-gas model with a non-trivial loop fugacity (Nienhuis 1982, Karowski and Rys 1986). Substituting (2) into the formula for the conformal dimensions (1), we find only a few which can possibly describe fractals for all N (we require that $0 \leqslant \Delta(N) \leqslant 1$, i.e. $0 \leqslant D(N) \leqslant 2$). These dimensions are listed in table 3. We have also given the values of D for $N = 1$ ($q = 0$, resistor network), $N = \frac{3}{2}$ ($q = 1$, percolation, LA and SAW), $N = 2$ ($q = 2$, Ising model) and $N = \infty$ ($q = 4$, four-state Potts and XY models). Using the obvious inequalities,

$$1 \leqslant D_{\min} \leqslant D_{\rm EBB} \leqslant D_{\rm BB} \leqslant D_{\rm f} \leqslant 2$$

$$D_{\rm red} \leqslant D_{\min} \leqslant D_{\rm SAW} \leqslant D_{\rm BB}$$

$$D_{\min} \leqslant D_{\rm u} \leqslant D_{\rm H} \leqslant D_{\rm f}$$

$$D_{\rm SAW} \leqslant D_{\rm LA}$$

and the conjectured values for $N = \frac{3}{2}$, we can uniquely assign a dimension for arbitrary N to each subfractal. This is indicated in the right-hand column of table 3. Several points are worth noting.

(i) When $N = 1$, $D_{\rm red} = 0$ and $D_{\rm f} = 2$. All sites are connected to the infinite cluster and therefore there are no red links.

(ii) $D_{\rm SAW}$ only equals the thermal exponent of the $O(n)$ model for $n = 0$ (de Gennes 1972). Since $y_{\rm t}(O(n)) < 1$ for $n > 1$, it cannot describe an interacting SAW for all n.

(iii) $D_{\rm SAW} = D_{\rm u}$ for all N. This is intuitively very appealing, since the unscreened perimeter of a cluster is both linear (i.e. non-branching) and self-avoiding.

(iv) There is no exponent which can be identified with the minimal path, because it would require $D_{\min} \geqslant 1$ for all N and $D_{\min} = \frac{17}{16}$ when $N = \frac{3}{2}$. Therefore we conjecture

Table 3. Possible conformal dimensions Δ from (3) which can correspond to fractals for arbitrary N. The fractal dimensions $D=2-2\Delta$ are listed for $N=1$ (resistor network), $N=\frac{3}{2}$ (percolation, SAW and lattice animals), $N=2$ (Ising) and $N=\infty$ (four-state Potts, XY).

	Conformal dimension $\Delta(N)$	Fractal dimension D $N=1$	$N=\frac{3}{2}$	$N=2$	$N=\infty$	
Δ_{11}	0	2	2	2	2	
Δ_{12}	$\frac{N+1}{2(2N-1)}$	0	$\frac{3}{4}$	1	$\frac{3}{2}$	y_t(Potts), D_{red}
Δ_{22}	$\frac{3}{8N(2N-1)}$	$\frac{5}{4}$	$\frac{7}{4}$	$\frac{15}{8}$	2	
Δ_{31}	$\frac{N-1}{N}$	2	$\frac{4}{3}$	1	0	$y_t(\mathrm{O}(n))$
Δ_{33}	$\frac{1}{N(2N-1)}$	0	$\frac{4}{3}$	$\frac{5}{3}$	2	
$\Delta_{2N,2N}$	$\frac{2N+1}{8N}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{11}{8}$	$\frac{3}{2}$	D_{SAW}, D_u
$\Delta_{2N-1,2N-1}$	$\frac{N-1}{2(2N-1)}$	2	$\frac{7}{4}$	$\frac{5}{3}$	$\frac{3}{2}$	D_H
$\Delta_{2N-2,2N-1}$	$\frac{(4N-2)^2-1}{8N(2N-1)}$	$\frac{5}{4}$	$\frac{17}{16}$	$\frac{13}{24}$	0	
$\Delta_{N,N}$	$\frac{N^2-1}{8N(2N-1)}$	2	$\frac{91}{48}$	$\frac{15}{8}$	$\frac{15}{8}$	y_h, D_f
$\Delta_{N,N-2}$	$\frac{9N^2-1}{8N(2N-1)}$	0	$\frac{19}{48}$	$\frac{13}{24}$	$\frac{7}{8}$	y_{h2}
$\Delta_{N-1,N}$	$\frac{3(3N-2)}{8(2N-1)}$	$\frac{5}{4}$	$\frac{17}{16}$	1	$\frac{7}{8}$	
$\Delta_{N-1,N-2}$	$\frac{N+2}{8(2N-1)}$	$\frac{5}{4}$	$\frac{25}{16}$	$\frac{5}{3}$	$\frac{15}{8}$	D_{LA}
$\Delta_{N-2,N-1}$	$\frac{(3N-2)^2-1}{8N(2N-1)}$	2	$\frac{25}{16}$	$\frac{11}{8}$	$\frac{7}{8}$	
$\Delta_{N-2,N-3}$	$\frac{(N+2)^2-1}{8N(2N-1)}$	0	$\frac{17}{16}$	$\frac{11}{8}$	$\frac{15}{8}$	

that

$$D_{\min}=\max(1, D_{red})$$

i.e. that the minimal path is essentially linear for $N<2$, while it is trapped by the red links for $N\geqslant 2$. This indicates that the minimal path does not correspond to a conformal field.

(v) $\Delta_{N,N-2}$ gives the second magnetic exponent (den Nijs 1983), y_{h2}. It is tempting to speculate that it might describe a first correction to the power law for the mass of the percolation cluster, i.e. that

$$M(R)=AR^D+BR^{y_{h2}}\ldots \tag{4}$$

(vi) The Sapoval *et al* (1985) conjecture

$$D_H = 1 + D_{red}$$

holds for $N = \frac{3}{2}$ but not for other values of N. Therefore it probably lacks deeper significance.

We have introduced the notion of conformal invariance into fractal physics. It appears that this approach is fruitful and yields a number of exact predictions. Unfortunately, the backbone dimension disagrees with numerical results, and possibly the lattice animal dimension does so as well. Despite these discrepancies, we think that the fractal dimensionalities fit rather well into the magic formula (3). We have managed to identify fractals for all x except $x = 7$ ($D = \frac{51}{48}$) and $x = 9$ ($D = \frac{19}{48}$). If these x also describe fractals, they must be of topological dimension 0, i.e. disconnected. For $x > 10$, (3) probably gives correction-to-scaling exponents, defined in analogy with (4).

There is one difference between the backbone and the minimal path on the one hand, and the other subsets on the other. To define the former, one must single out a preferred direction, while the definition of the latter can be made isotropically. This may be a reason for the apparent failure of conformal invariance to predict these dimensions.

It would be interesting to perform numerical experiments for $N \neq \frac{3}{2}$. Unfortunately, CWP is extremely difficult to simulate, because the cluster fugacity introduces long-range interactions. However, as argued by Larsson (1986), the Coniglio-Klein (1980) droplet construction can probably be viewed as a simulation of CWP. Thus, by measuring the subsets of droplets for $q = 2$, 3 and 4 ($N = 2$, 3 and ∞), it should be possible to check the conjectures of table 3.

We have only considered static properties of the cluster. Dynamic exponents like the spectral dimension and hybrid models like SAW on the cluster intuitively seem to require further input. As mentioned above, it also appears that the backbone does not fit our prediction.

In conclusion, it appears that, in two dimensions, conformal invariance is capable of satisfying Stanley's (1984) request for a simple principle governing the fractal dimensions. However, in this work we have only played games with Kac's formula (1), without any deeper understanding. It would be of great interest to understand what fractal is connected to what conformal dimension from fundamental principles.

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